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Reduced phase space: quotienting procedure for gauge theories

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Abstract. We present a reduction procedure for gauge theories based on quotienting out the kernel of the presymplectic form in configuration–velocity space. Local expressions for a basis of this kernel are obtained using phase-space procedures; the obstructions to the formulation of the dynamics in the reduced phase space are identified and circumvented. We show that this reduction procedure is equivalent to the standard Dirac method as long as the Dirac conjecture holds: that the Dirac Hamiltonian, containing the primary first-class constraints, with their Lagrange multipliers, can be enlarged to an extended Dirac Hamiltonian which includes all first-class constraints without any change of the dynamics. The quotienting procedure is always equivalent to the extended Dirac theory, even when it differs from the standard Dirac theory. The differences occur when there are ineffective constraints, and in these situations we conclude that the standard Dirac method is preferable—at least for classical theories. An example is given to illustrate these features, as well as the possibility of having phase-space formulations with an odd number of physical degrees of freedom.

1. Introduction

The dynamics of gauge theories is a very wide area of research because many fundamental physical theories are gauge theories. The basic ingredients are the variational principle, which derives the dynamics out of variations of an action functional, and the gauge principle, which is the driving principle for determining interactions based on a Lie group of internal symmetries. The gauge freedom exhibited by the complete theory becomes a redundancy in the physical description. The classical treatment of these systems was pioneered by Dirac (1950, 1964), Bergmann (1949), and Bergmann and Goldberg (1955). Dirac's classical treatment in phase space (the cotangent bundle for configuration space) has been shown (Gotay and Nester 1979, 1980, Batlle *et al* 1986) to be equivalent to the Lagrangian formulation in configuration–velocity space (the tangent bundle). One ends up with a constrained dynamics with some gauge degrees of freedom. One may choose, as is customary in many approaches (Pons and Shepley 1995), to introduce new constraints in the formalism to eliminate these unwanted—spurious—degrees of freedom. This is the gauge fixing procedure.

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There are approaches other than gauge fixing. For instance, the method of Faddeev and Jackiw (1993) and Jackiw (1995) is to attempt to reduce the system to its physical degrees of freedom by a process of directly substituting the constraints into the canonical Lagrangian. It has been proved (García and Pons 1997) that, as long as ineffective constraints—functions that vanish in the constraint surface and whose differentials also vanish there—are not present, the Faddeev–Jackiw method is equivalent to Dirac's.

A reduction procedure (Abraham and Marsden 1978, Sniatycki 1974, Lee and Wald 1990) which seems to be particularly appealing from a geometric point of view consists in quotienting out the kernel of the presymplectic form in configuration–velocity space in order to get a reduced space, the putative physical space, with a deterministic dynamics in it, that is, without gauge degrees of freedom. One must be careful that these techniques do not change the physics, for example by dropping degrees of freedom, and that they are applicable in all situations of physical interest. For example, we know of no treatment of this technique which applies to the important case when there are secondary constraints—one purpose of this paper is to provide this treatment.

In this paper we present a complete reduction method based on quotienting out the kernel of the presymplectic form. We establish a systematic algorithm and prove its equivalence with Dirac's method, but only so long as ineffective constraints do not appear. Our procedure turns out to be equivalent to Dirac's extended method, which enlarges the Hamiltonian by including all first-class constraints. It differs from the ordinary Dirac method (supplemented by gauge fixing) when ineffective constraints occur. Since the ordinary Dirac method is equivalent to the Lagrangian formalism, it is to be preferred in classical models.

We will consider Lagrangians with gauge freedom. Thus they must be singular: the Hessian matrix of the Lagrangian, consisting of its second partial derivatives with respect to the velocities, is singular or, equivalently, the Legendre transformation from configuration–velocity space to phase space is not locally invertible. Singular also means that the pullback under this map of the canonical form ω from phase space to configuration–velocity space is singular.

In order to proceed, we first compute, in section 2, in a local coordinate system, a basis for the kernel of the presymplectic form. Our results will be in general local; global results could be obtained by assuming the Lagrangian to be almost regular (Gotay and Nester 1980). In section 3, we will single out the possible problems in formulating the dynamics in the reduced space obtained by quotienting out this kernel. In section 4 we will solve these problems and will compare our results with the classical Dirac method. It proves helpful to work in phase space here, and we end up with a reduced phase space complete with a well defined symplectic form. In section 5 we illustrate our method with a simple example (which contains ineffective constraints). We draw our conclusions in section 6.

2. The kernel of the presymplectic form

We start with a singular Lagrangian $L(q^i, \dot{q}^i)$ $(i = 1, \ldots, N)$. The functions

$$\hat{p}_i(q, \dot{q}) := \partial L / \partial \dot{q}^i$$

are used to define the Hessian $W_{ij} = \partial \hat{p}_i / \partial \dot{q}^j$, a singular matrix that we assume has a constant rank $N - P$. The Legendre map $\mathcal{F}L$ from configuration–velocity space (the tangent bundle) TQ to phase space T^*Q, defined by $p_i = \hat{p}_i(q, \dot{q})$, defines a constraint surface of dimension $2N - P$.

The initial formulation of the dynamics in TQ uses the Lagrangian energy

$$E_{\rm L} := \hat{p}_i \dot{q}^i - L$$

and $\boldsymbol{X}$, the dynamical vector field on TQ,

$$i_{\boldsymbol{X}}\boldsymbol{\omega}_{\mathrm{L}} = \boldsymbol{d}(E_{\mathrm{L}}) \tag{2.1}$$

where

$$\boldsymbol{\omega}_{\mathrm{L}} := \boldsymbol{d}q^s \wedge \boldsymbol{d}\hat{p}_s$$

is the pullback under the Legendre map of the canonical form $\boldsymbol{\omega} = \boldsymbol{d}q^s \wedge \boldsymbol{d}p_s$ in phase space. $\boldsymbol{\omega}_{\mathrm{L}}$ is a degenerate, closed 2-form, which is called the presymplectic form on TQ. In fact there is an infinite number of solutions for equation (2.1) if the theory has gauge freedom, but they do not necessarily exist everywhere (if there are Lagrangian constraints). $\boldsymbol{X}$ must obey the second-order condition for a function

$$\boldsymbol{X}q^i = \dot{q}^i \Longleftrightarrow \boldsymbol{X} = \dot{q}^s \frac{\partial}{\partial q^s} + A^s(q, \dot{q}) \frac{\partial}{\partial \dot{q}^s}$$

where A^s is partially determined by equation (2.1).

At first sight, the kernel of $\boldsymbol{\omega}_{\mathrm{L}}$ describes, in principle, the arbitrariness in the solutions $\boldsymbol{X}$ of equation (2.1). Therefore, it is tempting to think that in order to construct a physical phase space, we must just quotient out this kernel. The complete implementation of this procedure which we are presenting in this paper is, first, far from obvious and, second, as we will see, fraught with danger.

Let us first determine a basis for

$$\mathcal{K} := \mathrm{Ker}(\boldsymbol{\omega}_{\mathrm{L}})$$

in local coordinates. We look for vectors $\boldsymbol{Y}$ satisfying

$$i_{\boldsymbol{Y}}\boldsymbol{\omega}_{\mathrm{L}} = 0. \tag{2.2}$$

With the notation

$$\boldsymbol{Y} = \epsilon^k \frac{\partial}{\partial q^k} + \beta^k \frac{\partial}{\partial \dot{q}^k}$$

equation (2.2) implies

$$\epsilon^i W_{ij} = 0 \tag{2.3a}$$

$$\epsilon^i A_{ij} - \beta^i W_{ij} = 0 \tag{2.3b}$$

where

$$A_{ij} := \frac{\partial \hat{p}_i}{\partial q^j} - \frac{\partial \hat{p}_j}{\partial q^i}.$$

Since $\boldsymbol{W}$ is singular (this causes the degeneracy of $\boldsymbol{\omega}_{\mathrm{L}}$), it possesses null vectors. It is very advantageous to this end to use information from phase space. It is convenient to use a basis for these null vectors, γ^i_μ ($\mu = 1, \ldots, P$), which is provided from the knowledge of the P primary Hamiltonian constraints of the theory, $\phi^{(1)}_\mu$. Actually, one can take (Batlle *et al* 1986),

$$\gamma^i_\mu = \mathcal{F}L^*\left(\frac{\partial \phi^{(1)}_\mu}{\partial p_i}\right) = \frac{\partial \phi^{(1)}_\mu}{\partial p_i}(q, \hat{p}) \tag{2.4}$$

where $\mathcal{F}L^*$ stands for the pullback of the Legendre map $\mathcal{F}L : TQ \longrightarrow T^*Q$. According to equation (2.3a), ϵ^i will be a combination of these null vectors: $\epsilon^i = \lambda^\mu \gamma^i_\mu$. Notice that we presume that these primary constraints are chosen to be effective.

To have a solution for β^i we need, after contraction of equation (2.3b) with the null vectors γ^j_ν,

$$0 = \lambda^\mu \gamma^i_\mu A_{ij} \gamma^j_\nu = \lambda^\mu \mathcal{F}L^*\left(\frac{\partial \phi^{(1)}_\mu}{\partial p_i}\right)\left(\frac{\partial \hat{p}_i}{\partial q^j} - \frac{\partial \hat{p}_j}{\partial q^i}\right)\mathcal{F}L^*\left(\frac{\partial \phi^{(1)}_\nu}{\partial p_j}\right)$$

which is to be understood as an equation for the λ^μs. We then use the identity

$$\mathcal{F}L^*\left(\frac{\partial\phi_\mu^{(1)}}{\partial p_j}\right)\frac{\partial\hat{p}_j}{\partial q^i}+\mathcal{F}L^*\left(\frac{\partial\phi_\mu^{(1)}}{\partial q^i}\right)=0 \tag{2.5}$$

which stems from the fact that $\phi_\mu^{(1)}(q,\hat{p})$ vanishes identically; we get

$$0=\lambda^\mu\mathcal{F}L^*\left(\frac{\partial\phi_\mu^{(1)}}{\partial p_i}\frac{\partial\phi_\nu^{(1)}}{\partial q^i}-\frac{\partial\phi_\mu^{(1)}}{\partial q^j}\frac{\partial\phi_\nu^{(1)}}{\partial p_j}\right)=\lambda^\mu\mathcal{F}L^*(\{\phi_\nu^{(1)},\phi_\mu^{(1)}\}). \tag{2.6}$$

Condition (2.6) means that the combination $\lambda^\mu\phi_\mu^{(1)}$ must be first class. Let us split the primary constraints $\phi_\mu^{(1)}$ between first class $\phi_{\mu_1}^{(1)}$ and second class $\phi_{\mu_1'}^{(1)}$ at the primary level, and we presume that second-class constraints are second class everywhere on the constraint surface (more constraints may become second class if we include secondary, tertiary, etc, constraints). They satisfy

$$\{\phi_{\mu_1}^{(1)},\phi_\mu^{(1)}\}=pc \qquad \det|\{\phi_{\mu_1'}^{(1)},\phi_{\nu_1'}^{(1)}\}|\neq 0 \tag{2.7}$$

where pc stands for a generic linear combination of primary constraints. Equations (2.6) simply enforce

$$\lambda^{\mu_1'}=0.$$

Consequently, a basis for the ϵ^i will be spanned by γ_{μ_1}, so that

$$\epsilon^i=\lambda^{\mu_1}\gamma_{\mu_1}^i$$

for λ^{μ_1} arbitrary. Once ϵ^i is given, solutions for β^i will then be of the form

$$\beta^i=\lambda^{\mu_1}\beta_{\mu_1}^i+\eta^\mu\gamma_\mu^i$$

where the η^μ are arbitrary functions on TQ. We will now determine the $\beta_{\mu_1}^j$.

To compute $\beta_{\mu_1}^j$ it is again very convenient to use Hamiltonian tools. Consider any canonical Hamiltonian $H_{\rm c}$ (which is not unique), that is, one satisfying $E_{\rm L}=\mathcal{F}L^*(H_{\rm c})$. Since we know from the classical Dirac analysis that the first-class primary constraints $\phi_{\mu_1}^{(1)}$ may produce secondary constraints,

$$\phi_{\mu_1}^{(2)}=\{\pi_{\mu_1}^{(1)},H_{\rm c}\}$$

we compute (having in mind equation (2.3*b*))

$$\begin{aligned}\gamma_{\mu_1}^iA_{ij}+\frac{\partial\phi_{\mu_1}^{(2)}}{\partial p_i}(q,\hat{p})W_{ij}&=\gamma_{\mu_1}^iA_{ij}+\frac{\partial\phi_{\mu_1}^{(2)}}{\partial p_i}(q,\hat{p})\frac{\partial\hat{p}_i}{\partial\dot{q}_j}\\&=\gamma_{\mu_1}^iA_{ij}+\frac{\partial\mathcal{F}L^*(\phi_{\mu_1}^{(2)})}{\partial\dot{q}_j}=\gamma_{\mu_1}^iA_{ij}+\frac{\partial(K\phi_{\mu_1}^{(1)})}{\partial\dot{q}_j}\end{aligned} \tag{2.8}$$

where we have used the operator K defined (Batlle *et al* 1986, Gràcia and Pons 1989) by

$$Kf:=\dot{q}^i\mathcal{F}L^*\left(\frac{\partial f}{\partial q^i}\right)+\frac{\partial L}{\partial q^i}\mathcal{F}L^*\left(\frac{\partial f}{\partial p_i}\right).$$

This operator satisfies (Batlle *et al* 1986, Pons 1988)

$$Kf=\mathcal{F}L^*(\{f,H_{\rm c}\})+v^\mu(q,\dot{q})\mathcal{F}L^*(\{f,\phi_\mu^{(1)}\}) \tag{2.9}$$

where the functions v^μ are defined through the identities

$$\dot{q}^i=\mathcal{F}L^*(\{q^i,H_{\rm c}\})+v^\mu(q,\dot{q})\mathcal{F}L^*(\{q^i,\phi_\mu^{(1)}\}). \tag{2.10}$$

Property (2.9) implies, for our first-class constraints,

$$K\phi^{(1)}_{\mu_1} = \mathcal{F}L^*(\phi^{(2)}_{\mu_1})$$

which has been used in equation (2.8). Let us continue with equation (2.8)

$$\gamma^i_{\mu_1} A_{ij} + \frac{\partial(K\phi^{(1)}_{\mu_1})}{\partial \dot{q}_j} = -\mathcal{F}L^*\left(\frac{\partial\phi^{(1)}_{\mu_1}}{\partial q^j}\right) - \mathcal{F}L^*\left(\frac{\partial\phi^{(1)}_{\mu_1}}{\partial p_i}\right)\frac{\partial \hat{p}_j}{\partial q^i}$$
$$+\frac{\partial}{\partial \dot{q}^j}\left(\dot{q}^i\mathcal{F}L^*\left(\frac{\partial\phi^{(1)}_{\mu_1}}{\partial q^i}\right) + \frac{\partial L}{\partial q^i}\mathcal{F}L^*\left(\frac{\partial\phi^{(1)}_{\mu_1}}{\partial p_i}\right)\right) = W_{ij}K\frac{\partial\phi^{(1)}_{\mu_1}}{\partial p_i} \tag{2.11}$$

where we have omitted some obvious steps to produce the final result. We can read off from this computation the solutions for equation (2.3*b*):

$$\beta^j_{\mu_1} = K\frac{\partial\phi^{(1)}_{\mu_1}}{\partial p_j} - \mathcal{F}L^*\left(\frac{\partial\phi^{(2)}_{\mu_1}}{\partial p_j}\right). \tag{2.12}$$

Therefore, a basis for $\mathcal{K}$ is provided by

$$\boldsymbol{\Gamma}_\mu := \gamma^j_\mu \frac{\partial}{\partial \dot{q}^j} \tag{2.13a}$$

and

$$\boldsymbol{\Delta}_{\mu_1} := \gamma^j_{\mu_1}\frac{\partial}{\partial q^j} + \beta^j_{\mu_1}\frac{\partial}{\partial \dot{q}^j}. \tag{2.13b}$$

Vectors $\boldsymbol{\Gamma}_\mu$ in equation (2.13*a*) form a basis for $\mathrm{Ker}(T\mathcal{F}L)$, where $T\mathcal{F}L$ is the tangent map of $\mathcal{F}L$ (also often denoted by $\mathcal{F}L_*$). They also span the vertical subspace of $\mathcal{K}$: $\mathrm{Ker}(T\mathcal{F}L) = \mathrm{Ver}(\mathcal{K})$. This is a well known result (Cariñena *et al* 1988), but as far as we know equations (2.13*a*) and (2.13*b*) are the first explicit local expression for $\mathcal{K}$ itself.

All other results (Cariñena 1990), obtained on geometrical grounds, for $\mathcal{K}$ are obvious once the basis for this kernel is displayed, as it is in equations (2.13*a*) and (2.13*b*). For instance, it is clear that $\dim\mathcal{K} \leqslant 2\dim\mathrm{Ver}(\mathcal{K})$. Also, defining the vertical endomorphism

$$\boldsymbol{S} = \frac{\partial}{\partial \dot{q}^i} \otimes \boldsymbol{d}q^i$$

we have $\boldsymbol{S}(\mathcal{K}) \subset \mathrm{Ver}(\mathcal{K})$. The case when

$$\boldsymbol{S}(\mathcal{K}) = \mathrm{Ver}(\mathcal{K})$$

corresponds, in the Hamiltonian picture, to the case when all primary constraints are first class (indices $\mu =$ indices μ_1). These are the so-called type II Lagrangians (Cantrjin *et al* 1986). $\boldsymbol{S}(\mathcal{K}) = \emptyset$ corresponds to the case when all primary constraints are second class (indices $\mu =$ indices μ'_1).

Equation (2.13*a*) implies, for any function f on T^*Q,

$$\boldsymbol{\Gamma}_\mu(\mathcal{F}L^*(f)) = 0. \tag{2.14}$$

The corresponding equation for $\boldsymbol{\Delta}_{\mu_1}$ is

$$\boldsymbol{\Delta}_{\mu_1}(\mathcal{F}L^*(f)) = \mathcal{F}L^*(\{f, \phi^{(1)}_{\mu_1}\}). \tag{2.15}$$

Since we will make use of this property later, we now prove this result. The action of $\boldsymbol{\Delta}_{\mu_1}$ is

$$\boldsymbol{\Delta}_{\mu_1}(\mathcal{F}L^*(f)) = \gamma^j_{\mu_1}\left(\mathcal{F}L^*\left(\frac{\partial f}{\partial q^j}\right) + \frac{\partial \hat{p}_i}{\partial q^j}\mathcal{F}L^*\left(\frac{\partial f}{\partial p_i}\right)\right) + \beta^j_{\mu_1}W_{ji}\mathcal{F}L^*\left(\frac{\partial f}{\partial p_i}\right).$$

We use equations (2.3*b*), (2.4), and (2.5) to get

$$\mathbf{\Delta}_{\mu_1}(\mathcal{F}L^*(f)) = \mathcal{F}L^*\left(\frac{\partial\phi^{(1)}_{\mu_1}}{\partial p_j}\frac{\partial f}{\partial q^j}\right) - \mathcal{F}L^*\left(\frac{\partial\phi^{(1)}_{\mu_1}}{\partial q^j}\frac{\partial f}{\partial p_j}\right)$$
$$= \mathcal{F}L^*(\{f, \phi^{(1)}_{\mu_1}\}).$$

The commutation relations (Lie brackets) for the vectors in equations (2.13*a*) and (2.13*b*) are readily obtained, and we present these new results here for the sake of completeness. We introduce the notation

$$\{\phi_{\mu_1}, \phi_\mu\} = A^\nu_{\mu_1\mu}\phi_\nu$$
$$\{\phi_{\mu_1}, \phi_{\nu_1}\} = B^{\rho_1}_{\mu_1\nu_1}\phi_{\rho_1} + \tfrac{1}{2}B^{\rho\sigma}_{\mu_1\nu_1}\phi_\rho\phi_\sigma$$

(commutation of first-class constraints is also first class). We arrive at

$$[\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu] = 0 \tag{2.16a}$$
$$[\mathbf{\Gamma}_\mu, \mathbf{\Delta}_{\mu_1}] = \mathcal{F}L^*(A^\nu_{\mu_1\mu})\mathbf{\Gamma}_\nu \tag{2.16b}$$
$$[\mathbf{\Delta}_{\mu_1}, \mathbf{\Delta}_{\nu_1}] = \mathcal{F}L^*(B^{\rho_1}_{\nu_1\mu_1})\mathbf{\Delta}_{\rho_1} + v^{\delta'_1}\mathcal{F}L^*(B^{\rho\sigma'_1}_{\nu_1\mu_1}\{\phi_{\sigma'_1}, \phi_{\delta'_1}\})\mathbf{\Gamma}_\rho \tag{2.16c}$$

where the $v^{\delta'_1}$ are defined in equation (2.10). Observe that the number of vectors in equations (2.13*a*) and (2.13*b*) is even because $|\mu'_1| = |\mu| - |\mu_1|$ is the number of second-class primary constraints (at the primary level), which is even.

Because the algebra of $\mathcal{K}$ is closed, the action of $\mathcal{K}$ on TQ is an equivalence relation. We can form the quotient space $TQ/\mathcal{K}$ and the projection

$$\pi : TQ \longrightarrow TQ/\mathcal{K}.$$

$TQ/\mathcal{K}$ is endowed with a symplectic form obtained by quotienting out the null vectors of $\boldsymbol{\omega}_{\mathrm{L}}$ (that is, $\boldsymbol{\omega}_{\mathrm{L}}$ is projectable to $TQ/\mathcal{K}$). The space $TQ/\mathcal{K}$ is not necessarily the final physical space, however, because we have not yet formulated the dynamics of the system. We now turn to the question of the projectability of the Lagrangian energy.

3. Obstructions to the projectability of the Lagrangian energy

In order to project the dynamical equation (2.1) to $TQ/\mathcal{K}$, we need E_{L} to be projectable under π. However, in order for E_{L} to be projectable we must check whether it is constant on the orbits generated by $\mathcal{K}$ as defined by the vector fields of equations (2.13*a*) and (2.13*b*). Indeed $\mathbf{\Gamma}_\mu(E_{\mathrm{L}}) = 0$, but from equation (2.15)

$$\mathbf{\Delta}_{\mu_1}(E_{\mathrm{L}}) = -\mathcal{F}L^*(\phi^{(2)}_{\mu_1})$$

where

$$\phi^{(2)}_{\mu_1} := \{\phi^{(1)}_{\mu_1}, H_{\mathrm{c}}\}.$$

If $\mathcal{F}L^*(\phi^{(2)}_{\mu_1}) \neq 0$ for some μ_1, then $\phi^{(2)}_{\mu_1}$ is a secondary Hamiltonian constraint. As a side remark, note that in this case $\mathcal{F}L^*(\phi^{(2)}_{\mu_1})$ is a primary Lagrangian constraint. In fact it can be written (Batlle *et al* 1986) as

$$\mathcal{F}L^*(\phi^{(2)}_{\mu_1}) = [L]_i\gamma^i_{\mu_1}$$

where $[L]_i$ is the Euler–Lagrange functional derivative of L.

We see that there is an obstruction to the projectability of E_{L} to $TQ/\mathcal{K}$ as long as there exist secondary Hamiltonian constraints or equivalently if there exist Lagrangian constraints.

One way to remove this problem (Ibort and Marín-Solano 1992, Ibort *et al* 1993) is to use the co-isotropic embedding theorems (Gotay 1982, Gotay and Sniatycki 1981) and look for an extension of the tangent space possessing a regular Lagrangian that extends the singular one and leads to a consistent theory once the extra degrees of freedom are removed. This method is equivalent to Dirac's, but only if there are no secondary Hamiltonian constraints. However, if there are, which is precisely our case, the dynamics becomes modified and thus changes the original variational principle. Instead of using this technique we will try to preserve the dynamics.

4. Physical space

In the cases where secondary Hamiltonian constraints do exist (for instance, Yang–Mills and Einstein–Hilbert theories), we must find an alternative reduction of TQ in order to obtain the projectability of E_{L}.

The initial idea was to quotient out the orbits defined by equations (2.13*a*) and (2.13*b*). Since $\boldsymbol{\Gamma}_\mu(E_{\mathrm{L}}) = 0$ we can at least quotient out the orbits defined by equation (2.13*a*). However, this quotient space, $TQ/\mathrm{Ker}(T\mathcal{F}L)$, is already familiar to us: it is isomorphic to the surface M_1 defined by the primary constraints in T^*Q. In fact, if we define π_1 as the projection

$$\pi_1 : TQ \longrightarrow TQ/\mathrm{Ker}(T\mathcal{F}L)$$

we have the decomposition of the Legendre map $\mathcal{F}L = i_1 \circ \pi_1$, where

$$i_1 : \frac{TQ}{\mathrm{Ker}(T\mathcal{F}L)} \longrightarrow T^*Q$$

with

$$i_1\left(\frac{TQ}{\mathrm{Ker}(T\mathcal{F}L)}\right) = M_1.$$

Now we can take advantage of working in $M_1 \subset T^*Q$. Let us project our original structures on TQ into M_1. Consider the vector fields $\boldsymbol{\Delta}_{\mu_1}$. Equation (2.15) tells us that the vector fields $\boldsymbol{\Delta}_{\mu_1}$ are projectable to M_1 and that their projections are just $\{-, \phi^{(1)}_{\mu_1}\}$. In fact these vector fields $\{-, \phi^{(1)}_{\mu_1}\}$ are vector fields of T^*Q, but they are tangent to M_1 because $\phi^{(1)}_{\mu_1}$ are first class (among the primary constraints defining M_1). Incidentally, note that the vector fields $\{-, \phi^{(1)}_{\mu'_1}\}$ associated with the second-class primary constraints in T^*Q are not tangent to M_1.

Formulation in M_1 of the dynamics corresponding to equation (2.1) uses the pre-symplectic form $\boldsymbol{\omega}_1$ defined by $\boldsymbol{\omega}_1 = i_1^*\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the canonical form in phase space, and the Hamiltonian H_1 defined by $H_1 = i_1^* H_{\mathrm{c}}$, with H_{c} such that $\mathcal{F}L^*(H_{\mathrm{c}}) = E_{\mathrm{L}}$. The dynamic equation in M_1 will be

$$i_{\boldsymbol{X}_1}\boldsymbol{\omega}_1 = \boldsymbol{d}H_1. \tag{4.1}$$

The null vectors for $\boldsymbol{\omega}_1$ are $\{-, \phi^{(1)}_{\mu_1}\}$ (more specifically, their restriction to M_1). (This result is general: the kernel of the pullback of the symplectic form to a constraint surface in T^*Q is locally spanned by the vectors associated, through the Poisson bracket, with the first-class constraints among the constraints which define the surface.) To project the dynamics of equation (4.1) to the quotient of M_1 by the orbits defined by $\{-, \phi^{(1)}_{\mu_1}\}$

$$\mathcal{P}_1 := \frac{M_1}{(\{-, \phi^{(1)}_{\mu_1}\})} \tag{4.2}$$

we need the projectability of H_1 to this quotient manifold. To check this requirement it is better to work in T^*Q. Then projectability of H_1 to $\mathcal{P}_1$ is equivalent to requiring that $\{\phi^{(1)}_{\mu_1}, H_c\}|_{M_1} = 0$.

Here lies the obstruction we met in the previous section, for it is possible that $\{\phi^{(1)}_{\mu_1}, H_c\}|_{M_1} \neq 0$ for some constraints $\phi^{(1)}_{\mu_1}$. Let us assume that this is the case. As we did before, we define

$$\phi^{(2)}_{\mu_1} := \{\phi^{(1)}_{\mu_1}, H_c\}.$$

These constraints may not be independent, some of them may vanish on M_1, and some previously first-class constraints may become second class. Those that do not vanish are secondary constraints and allow us to define the new surface $M_2 \subset M_1$ (we define the map $i_2 : M_2 \longrightarrow M_1$) by $\phi^{(2)}_{\mu_1} = 0$.

We can now form the projection of $H_2 := i_2^* H_1$ to $M_2/(\{-, \phi^{(1)}_{\mu_1}\})$, but the projection of $\boldsymbol{\omega}_2 := i_2^* \boldsymbol{\omega}_1$ can still be degenerate in this quotient space, since $\boldsymbol{\omega}_2$ may have acquired new null vectors (and may have lost some of the old ones). In fact, once all constraints are expressed in effective form, $\mathrm{Ker}(\boldsymbol{\omega}_2)$ is generated under the Poisson bracket associated with $\boldsymbol{\omega}$ by the subset of effective constraints that are first class with respect to the whole set of constraints defining M_2. If there is a piece in this kernel that was not present in $\mathrm{Ker}(\boldsymbol{\omega}_1)$, then new conditions for the projectability of H_2 will appear.

The dynamic equation in M_2 is

$$i_{X_2}\boldsymbol{\omega}_2 = \boldsymbol{d}H_2. \tag{4.3}$$

It is still convenient to work with structures defined in T^*Q. Suppose that $\phi^{(2)}_{\mu_2}$ is any secondary, first-class, effective constraint in M_2; therefore, $\{-, \phi^{(2)}_{\mu_2}\} \in \mathrm{Ker}(\boldsymbol{\omega}_2)$ but $\{-, \phi^{(2)}_{\mu_2}\} \notin \mathrm{Ker}(\boldsymbol{\omega}_1)$. The new projectability condition for H_2 induced by $\phi^{(2)}_{\mu_2}$ is

$$\{\phi^{(2)}_{\mu_2}, H_c\}|_{M_2} = 0.$$

This means that we might find new constraints if this condition is not satisfied. A new surface M_3 will appear, and a new kernel for a new $\boldsymbol{\omega}_3$ should be quotiented out, and so on. We will not go further because we are just reproducing Dirac's algorithm in phase space (Dirac 1950, 1964, Batlle *et al* 1986, Gotay *et al* 1978). We do have a shift of language, however: what in Dirac's standard algorithm is regarded as a condition for the Hamiltonian vector field to be tangent to the constraint surface is regarded here as a projectability condition for the Hamiltonian to a quotient space.

To summarize: the constraint surface M_1 is defined by the primary constraints $\phi^{(1)}_{\mu}$, a subset of which are the first-class constraints $\phi^{(1)}_{\mu_1}$. These first-class constraints are used in the formation of the quotient space

$$\mathcal{P}_1 = \frac{M_1}{\{-, \phi^{(1)}_{\mu_1}\}}.$$

The projectability condition for H_1 (the pullback of H_c to M_1) to $\mathcal{P}_1$ may be expressed as the condition $\{H_c, \phi^{(1)}_{\mu_1}\}|_{M_1} = 0$. If this condition holds, we have found the final physical space. If it does not, there are new, secondary constraints $\phi^{(2)}_{\mu_1}$, and these constraints along with the initial set of primary constraints $\phi^{(1)}_{\mu}$ are used to define a constraint surface M_2. Among the set of constraints used to define M_2 are first-class constraints, including a subset of the first-class constraints associated with M_1, which we denote $\phi^{(1)}_{\mu_2}$, and a subset of the secondary constraints, which we denote $\phi^{(2)}_{\mu_2}$. These first-class constraints are used in the formulation of the quotient space

$$\mathcal{P}_2 := \frac{M_2}{(\{-, \phi^{(1)}_{\mu_2}\}, \{-, \phi^{(2)}_{\mu_2}\})}.$$

Again we must require projectability of the Hamiltonian; eventually, the final phase space is

$$\mathcal{P}_f := \frac{M_f}{(\{-, \phi^{(1)}_{\mu_f}\}, \{-, \phi^{(2)}_{\mu_f}\}, \ldots, \{-, \phi^{(k)}_{\mu_f}\})} \tag{4.4}$$

where $\phi^{(n)}_{\mu_f}$ are the final first-class n-ary constraints, all of which are taken in effective form. $\mathcal{P}_f$ is endowed with a symplectic form which is the projection of the form $\boldsymbol{\omega}_f$ in M_f, which is the final constraint surface. The dimension of $\mathcal{P}_f$ is $2N - M - P_f$, where N is the dimension of the initial configuration space, M is the total number of constraints, and P_f is the number of final first-class constraints. Observe that we end up with the standard counting of degrees of freedom for constrained dynamical systems: first-class constraints eliminate two degrees of freedom each, whereas second-class constraints eliminate only one each. The final result is an even number because the number of second-class constraints is even.

In order to use the technique we have presented, the constraints are presumed to be effective (for example, see equation (2.4))—if ineffective constraints occur, they can always be made effective for use with this technique; in that sense, the technique is actually geometrical. One might ask whether such modification of ineffective constraints can cause problems. It turns out that if ineffective constraints occur, then their presence may modify the gauge fixing procedure used in conjunction with the original Dirac method in such a way that the counting of degrees of freedom differs from that presented above. In the next section we discuss a simple example that shows the difference between Dirac's original treatment, supplemented by gauge fixing, and the quotienting method we have outlined here, which corresponds to Dirac's extended method.

Dirac's extended method, which is equivalent to the one we have presented here, produces a final phase space which is always even-dimensional. Dirac's original procedure, supplemented by gauge fixing, has the superiority of being equivalent to the Lagrangian variational principle. Therefore, in spite of the fact that this latter method may result in a system with an odd number of degrees of freedom (as in the example in the following section), it is to be preferred for classical models.

5. Example

Consider the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2z}\dot{y}^2 \tag{5.1}$$

where $z \neq 0$. The Noether gauge transformations are

$$\delta x = 0 \qquad \delta y = \frac{\epsilon\dot{y}}{z} \qquad \delta z = \dot{\epsilon}$$

where ϵ is an arbitrary function.

First, we analyse this system from a Lagrangian point of view. The equations of motion are

$$\ddot{x} = 0 \qquad \dot{y} = 0. \tag{5.2}$$

The z variable is completely arbitrary and is pure gauge. These equations define a system with three degrees of freedom in tangent space, parametrized by $x(0), \dot{x}(0), y(0)$. Notice that the gauge transformation δy vanishes on shell, so y is a weakly gauge invariant quantity.

Let us now analyse this system using Dirac's method. The Dirac Hamiltonian is

$$H_D = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}zp_y^2 - \lambda p_z \tag{5.3}$$

where λ is the Lagrange multiplier for the primary constraint $p_z = 0$. The stabilization of $p_z = 0$ gives the ineffective constraint $p_y^2 = 0$, and the algorithm stops here. The gauge generator (Batlle *et al* 1989, Pons *et al* 1997) is

$$G = \dot{\epsilon} p_z + \frac{\epsilon}{2} p_y^2 \tag{5.4}$$

with ϵ an arbitrary function of time.

The gauge fixing procedure (Pons and Shepley 1995) has in general two steps. The first is to fix the dynamics and the second is to eliminate redundant initial conditions. Here, to fix the dynamics we can introduce the general gauge fixing $z - f(t) = 0$ for f arbitrary. Stability of this condition under the gauge transformations sets $\dot{\epsilon} = 0$. Since the coefficient of ϵ in G is ineffective, it does not change the dynamical trajectories, and so the gauge fixing is complete. Notice that this violates the standard lore, for we have two first-class constraints, $p_z = 0$ and $p_y = 0$, but only one gauge fixing. This totals three constraints that reduce the original six degrees of freedom to three, $x(0)$, $p_x(0)$, $y(0)$, the same as in the Lagrangian picture.

Instead, if we apply the method of quotienting out the kernel of the presymplectic form, we get as a final reduced phase space

$$\mathcal{P}_f = \frac{M_2}{(\{-, p_z\}, \{-, p_y\})}$$

where M_2 is the surface in phase space defined by $p_z = 0$, $p_y = 0$. We have $\boldsymbol{\omega}_2 = \boldsymbol{d}x \wedge \boldsymbol{d}p_x$ and $H_2 = \frac{1}{2} p_x^2$. The dimension of $\mathcal{P}_f$ is two.

This result, which is different from that using Dirac's method, matches the one obtained with the extended Dirac Hamiltonian, where all final first-class constraints (in effective form) are added with Lagrange multipliers to the canonical Hamiltonian. Dirac's conjecture was that the original Dirac theory and the extended one were equivalent. We conclude that when Dirac's conjecture holds, the method of quotienting out the kernel is equivalent to Dirac's, whereas if Dirac's conjecture fails, it is equivalent to the extended Dirac's formalism.

6. Conclusions

In summary, we have the following.

(1) We have obtained a local basis for $\mathcal{K} = \mathrm{Ker}(\boldsymbol{\omega}_{\mathrm{L}})$ in configuration–velocity space for any gauge theory. This is new and allows for trivial verifications of the properties of $\mathcal{K}$ given in the literature. To obtain these results it has been particularly useful to rely on Hamiltonian methods.

(2) We have obtained as the final reduced phase space the quotient of the final Dirac constraint surface in the canonical formalism by the integral surface generated by the final first-class constraints in effective form. We find the constraint surface (M_f in equation (4.4)) through a projectability requirement on the Lagrangian energy (or equivalently, on the Hamiltonian) rather than through imposing tangency conditions on the Hamiltonian flows. Let us emphasize this point: we do not talk of stabilization of constraints but rather projectability of structures which are required to formulate the dynamics in a reduced physical phase space.

(3) We have compared our results with Dirac's procedure. An agreement exists in all the cases when no ineffective Hamiltonian constraints appear in the formalism. If there are ineffective constraints whose effectivization is first class, then our results disagree with Dirac's, and it turns out that the quotienting algorithm agrees with the extended Dirac formalism. When there are disagreements, the origin is in the structure of the gauge generators. Sometimes the gauge generators contain pieces that are ineffective constraints, and they, contrary to the usual case, do not call for any gauge fixing. Essentially, the variables that are canonically conjugate to

these first-class ineffective constraints are weakly (on shell) gauge invariant. The quotienting reduction method, as well as Dirac's extended formulation, eliminates these variables and yields a phase space whose variables are strictly (on- and off-shell) gauge invariant. This is the difference with Dirac's original method, supplemented with gauge fixing, which is able to retain the weakly gauge invariant quantities. For this reason we feel that this latter technique is superior to the quotienting algorithm in these circumstances—at least for classical models.

(4) We have produced a simple example that illustrates the failure of Dirac's conjecture in the presence of ineffective constraints. This example also shows that, in Dirac's analysis, it is possible to have Hamiltonian formulations with an odd number of physical degrees of freedom. We must remark that in Dirac's approach (supplemented with gauge fixing) it is not always true that every first-class constraint eliminates two degrees of freedom: this does not happen if there are first-class constraints that appear in the stabilization algorithm in ineffective form.

(5) It is worth mentioning that other reduction techniques, specifically the Faddeev and Jackiw method, may also fail to reproduce Dirac's theory (García and Pons 1998) if the formalism contains ineffective constraints.

(6) Of course, one should not forget quantum mechanics. The canonical approach to quantum mechanics involves a (non-singular) symplectic form (Isham 1984). In this method, it is therefore required that phase space be even-dimensional. This argument would tend to favour the quotienting algorithm. However, it may be that other approaches to quantum mechanics, possibly the path integration approach, do not need such a requirement. In any case, it is not strictly necessary that a model which is acceptable as a classical model be quantizable. It is for these reasons that we say that an approach to Hamiltonian dynamics which results in a phase-space picture equivalent to the tangent space picture—the original Dirac method supplemented with gauge fixing—is favoured for classical models.

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References

Abraham R and Marsden J E 1978 *Foundations of Mechanics* 2nd edn (Reading, MA: Benjamin-Cummings)

Batlle C, Gomis J, Gràcia X and Pons J M 1989 Neother's theorem and gauge transformations: application to the bosonic string and CP_2^{n-1} *J. Math. Phys.* **30** 1345–50

Batlle C, Gomis J, Pons J M and Roman N 1986 Equivalence between the Lagrangian and Hamiltonian formalism for constrained systems *J. Math. Phys.* **27** 2953–62

Bergmann P G 1949 Non-linear field theories *Phys. Rev.* **75** 680–5

Bergmann P G and Goldberg I 1955 Dirac bracket transformations in phase space *Phys. Rev.* **98** 531–8

Cantrjin F, Cariñena J F, Crampin M and Ibort L A 1986 Reduction of degenerate Lagrangian systems *J. Geom. Phys* **3** 353–400

Cariñena J F 1990 Theory of singular Lagrangians *Forstsch. Phys.* **38** 641–79 and references therein

Cariñena J F, López C and Román-Roy N 1988 Origin of the Lagrangian constraints and their relation with the Hamiltonian formulation *J. Math. Phys.* **29** 1143–9

Dirac P A M 1950 Generalized Hamiltonian dynamics *Can. J. Math.* **2** 129–48

——1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University Press)

Faddeev L and Jackiw R 1993 Hamiltonian reduction of unconstrained and constrained systems *Phys. Rev. Lett.* **60** 1692–4

García J A and Pons J M 1997 Equivalence of Faddeev–Jackiw and Dirac approaches for gauge theories *Int. J. Mod. Phys.* A **12** 451–64

——1998 Faddeev–Jackiw approach to gauge theories and ineffective constraints *Int. J. Mod. Phys.* A, to be published

Gotay M 1982 On coisotropic imbeddings of presymplectic manifolds *Proc. Am. Math. Soc.* **84** 111–14

Gotay M J and Nester J M 1979 Presymplectic Lagrangian systems I: the constraint algorithm and the equivalence theorem *Ann. Inst. H. Poincaré* A **30** 129–42

——1980 Presymplectic Lagrangian systems II: the second-order equation problem *Ann. Inst. H. Poincaré* A **32** 1–13

Gotay M J, Nester J M and Hinds G 1978 Presymplectic manifolds and the Dirac–Bergmann theory of constraints *J. Math. Phys.* **19** 2388–99

Gotay M and Sniatycki J 1981 On the quantization of presymplectic dynamical systems via coisotropic imbeddings *Commun. Math. Phys.* **82** 377–89

Gràcia X and Pons J M 1989 On an evolution operator connecting Lagrangian and Hamiltonian formalisms *Lett. Math. Phys.* **17** 175–80

Ibort L A, Landi G, Marín-Solano J and Marmo G 1993 On the inverse problem of Lagrangian supermechanics *Int. J. Mod. Phys.* A **8** 3565–76

Ibort L A and Marín-Solano J 1992 A geometric classification of Lagrangian functions and the reduction of evolution space *J. Phys. A: Math. Gen.* **25** 3353–67

Isham C J 1984 Topological and global aspects of quantum theory *Relativité, Groupes, et Topologie II* ed B S DeWitt and R Stora (Amsterdam: North-Holland) pp 1059–290

Jackiw R 1995 (Constrained) quantization without tears *Proc. 2nd Workshop on Constraints Theory and Quantization Methods (Montepulciano, 1993)* (Singapore: World Scientific) pp 163–75

Lee J and Wald R M 1990 Local symmetries and constraints *J. Math. Phys.* **31** 725–43

Pons J M 1988 New relations between Hamiltonian and Lagrangian constraints *J. Phys. A: Math. Gen.* **21** 2705–15

Pons J M, Salisbury D C and Shepley L C 1997 Gauge transformations in the Lagrangian and Hamiltonian formalisms of generally covariant systems *Phys. Rev.* D **55** 658–68

Pons J M and Shepley L C 1995 Evolutionary laws, initial conditions and gauge fixing in constrained systems *Class. Quantum Grav.* **12** 1771–90

Sniatycki J 1974 Dirac brackets in geometric dynamics *Ann. Inst. H. Poincaré* A **20** 365–72